Transmission Lines and Power Flow Analysis

Dr. Greg Mowry
Annie Sebastian
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The UST Mission

Inspired by Catholic intellectual tradition, the University of St. Thomas educates students to be morally responsible leaders who think critically, act wisely, and work skillfully to advance the common good.

http://www.stthomas.edu/mission/

Since civilization depends on power, power is for the ‘common good’!!
Outline

I. Background material

II. Transmission Lines (TLs)

III. Power Flow Analysis (PFA)
I. Background Material
In a network (power system) there are 6 basic electrical quantities of interest:

1. Current \( i(t) = \frac{dq}{dt} \)

2. Voltage \( v(t) = \frac{d\varphi}{dt} \) (FL)

3. Power \( p(t) = v(t) i(t) = \frac{dw}{dt} = \frac{dE}{dt} \)
6 basic electrical quantities continued:

4. Energy (work) \[ w(t) = \int_{-\infty}^{t} p(\tau) d\tau = \int_{-\infty}^{t} v(\tau)i(\tau) d\tau \]

5. Charge (q or Q) \[ q(t) = \int_{-\infty}^{t} i(\tau) d\tau \]

6. Flux \[ \varphi(t) = \int_{-\infty}^{t} \nu(\tau) d\tau \]
A few comments:

- Voltage (potential) has potential-energy characteristics and therefore needs a reference to be meaningful; e.g., ground \( \equiv 0 \) volts.

- I will assume that pretty much everything we discuss & analyze today is linear until we get to PFA.
Networks & Power Systems

Linear Time Invariant (LTI) systems:

- Linearity:

  If \( y_1 = f(x_1) \) \& \( y_2 = f(x_2) \)

  Then \( \alpha \ y_1 + \beta \ y_2 = f(\alpha \ x_1 + \beta \ x_2) \)
Networks & Power Systems

- Implications of LTI systems:
  - Scalability: if $y = f(x)$, then $\alpha y = f(\alpha x)$
  - Superposition & scalability of inputs holds
  - Frequency invariance: $\omega_{input} = \omega_{output}$
ACSS and LTI systems:

Suppose: \( v_{\text{in}}(t) = V_{\text{in}} \cos(\omega t + \theta) \)

Since \( \omega_{\text{input}} = \omega_{\text{output}} \), the only thing a linear system can do to an input is:

- Change the input amplitude: \( V_{\text{in}} \rightarrow V_{\text{out}} \)
- Change the input phase: \( \theta_{\text{in}} \rightarrow \theta_{\text{out}} \)
ACSS and LTI systems:

These LTI characteristics along with Euler’s theorem

\[ e^{j\omega t} = \cos(\omega t) + j \sin(\omega t), \]

allow \( v_{in}(t) = V_i \cos(\omega t + \theta) \) to be represented as

\[ v_{in}(t) = V_i \cos(\omega t + \theta) \rightarrow V_i e^{j\theta} = V_i \angle \theta \]
Networks & Power Systems

➢ In honor of Star Trek,

\[ V_i e^{j \theta} = V_i \angle \theta \]

is called a ‘Phasor’
$V_i e^{j\theta} = V_i \angle \theta$
ACSS and LTI systems:

- Since time does not explicitly appear in a phasor, an LTI system in ACSS can be treated like a DC system on steroids with $j$

- ACSS also requires impedances; i.e. AC resistance
Comments:

- Little, if anything, in ACSS analysis is gained by thinking of $j = \sqrt{-1}$ (that is algebra talk)

- Rather, since Euler Theorem looks similar to a 2D vector

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

$$\bar{u} = a \hat{x} + b \hat{y}$$
\[ e^{j90^\circ} = \cos(90^\circ) + j \sin(90^\circ) = j \]

Hence in ACSS analysis, think of \( j \) as a \( 90^\circ \) rotation
Operators

\[ v(t) \quad \longleftrightarrow \quad V = V \angle \phi \]

\[ \frac{dv}{dt} \quad \longleftrightarrow \quad j\omega V \]

\[ \int v \, dt \quad \longleftrightarrow \quad \frac{V}{j\omega} \]
### Ohm’s Approximation (Law)

#### Summary of voltage-current relationship

<table>
<thead>
<tr>
<th>Element</th>
<th>Time domain</th>
<th>Frequency domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>$v = Ri$</td>
<td>$V = RI$</td>
</tr>
<tr>
<td>L</td>
<td>$v = L\frac{di}{dt}$</td>
<td>$V = j\omega LI$</td>
</tr>
<tr>
<td>C</td>
<td>$i = C\frac{dv}{dt}$</td>
<td>$V = \frac{I}{j\omega C}$</td>
</tr>
</tbody>
</table>
# Impedance & Admittance

Impedances and admittances of passive elements

<table>
<thead>
<tr>
<th>Element</th>
<th>Impedance</th>
<th>Admittance</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>$Z = R$</td>
<td>$Y = \frac{1}{R}$</td>
</tr>
<tr>
<td>L</td>
<td>$Z = j\omega L$</td>
<td>$Y = \frac{1}{j\omega L}$</td>
</tr>
<tr>
<td>C</td>
<td>$Z = \frac{1}{j\omega C}$</td>
<td>$Y = j\omega C$</td>
</tr>
</tbody>
</table>
The Impedance Triangle

\[ Z = R + j X \]
The Power Triangle (ind. reactance)

\[ S = P + jQ \]
Comments:

- $\phi_{\text{Power Triangle}} = \phi_{\text{Impedance Triangle}}$

- The $\cos(\phi)$ defines the Power Factor (pf)
Reminder of Useful Network Theorems

- KVL – Energy conservation  \( \sum v_{\text{Loop}} = 0 \)
- KCL – Charge conservation  \( \sum i_{\text{Node or bus}} = 0 \)
- Ohm’s Approximation (Law)
- Passive sign Convention (PSC)
Passive Sign Convection – Power System
Source

Load or Circuit Element

\[ P_{\text{Source}} = vi \]

\[ P_{\text{Load}} = vi \]

\[ P_{\text{Source}} = P_{\text{Load}} \]
Reminder of Useful Network Theorems

- **Pointing Theorem** \( S = V I^* \)

- **Thevenin Equivalent** – Any linear circuit may be represented by a voltage source and series Thevenin impedance

- **Norton Equivalent** – Any linear circuit may be represented by a current source and a shunt Thevenin impedance

\[ V_{TH} = I_N Z_{TH} \]
Transmission Lines and Power Flow Analysis

Dr. Greg Mowry
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II. Transmission Lines
Outline

- General Aspects of TLs
- TL Parameters: R, L, C, G
- 2-Port Analysis (a brief interlude)
- Maxwell’s Equations & the Telegraph equations
- Solutions to the TL wave equations
Transmission Lines (TLs)

- A TL is a major component of an electrical power system.

- The function of a TL is to transport power from sources to loads with minimal loss.
Transmission Lines

- In an AC power system, a TL will operate at some frequency $f$; e.g. 60 Hz.

- Consequently, $\lambda = \frac{c}{f}$

- The characteristics of a TL manifest themselves when $\lambda \sim L$ of the system.
USA example

- The USA is ~ 2700 mi. wide & 1600 mi. from north-to-south (4.35e6 x 2.57e6 m).

- For 60 Hz, \( \lambda = \frac{3e8}{60} = 5e6 \text{ m} \)

- Thus \( \lambda_{60\text{Hz}} \sim L_{USA} \) and consequently the continental USA transmission system will exhibit TL characteristics
Overhead TL Components

- Shield Wire
- Insulator
- Conductors
- Structures
- Right-of-Way
Overhead ACSR TL Cables
Some Types of Overhead ACSR TL Cables

TL Parameters
Model of an Infinitesimal TL Section
General TL Parameters

a. **Series Resistance** – accounts for Ohmic ($I^2R$ losses)

b. **Series Impedance** – accounts for series voltage drops
   - Resistive
   - Inductive reactance

c. **Shunt Capacitance** – accounts for Line-Charging Currents

d. **Shunt Conductance** – accounts for $V^2G$ losses due to leakage currents between conductors or between conductors and ground.
Power TL Parameters

- **Series Resistance**: related to the physical structure of the TL conductor over some temperature range.

- **Series Inductance & Shunt Capacitance**: produced by magnetic and electric fields around the conductor and affected by their geometrical arrangement.

- **Shunt Conductance**: typically very small so neglected.
## TL Design Considerations

<table>
<thead>
<tr>
<th>Design Considerations</th>
<th>Responsibilities</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Electrical Factors</strong></td>
<td>1. Dictates the size, type and number of bundle conductors per phase.</td>
</tr>
<tr>
<td></td>
<td>2. Responsible for number of insulator discs, vertical or v-shaped arrangement,</td>
</tr>
<tr>
<td></td>
<td>phase to phase clearance and phase to tower clearance to be used</td>
</tr>
<tr>
<td></td>
<td>3. Number, Type and location of shield wires to intercept lightning strokes.</td>
</tr>
<tr>
<td></td>
<td>4. Conductor Spacing's, Types and sizes</td>
</tr>
<tr>
<td><strong>Mechanical Factors</strong></td>
<td>Focuses on Strength of the conductors, insulator strings and support structures</td>
</tr>
<tr>
<td><strong>Environmental Factors</strong></td>
<td>Include Land usage, and visual impact</td>
</tr>
<tr>
<td><strong>Economic Factors</strong></td>
<td>Technical Design criteria at lowest overall cost</td>
</tr>
</tbody>
</table>
Transmission Line Components

<table>
<thead>
<tr>
<th>Components</th>
<th>Made of</th>
<th>Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conductors</td>
<td>Aluminum replaced copper</td>
<td>ACSR - Aluminum Conductor Steel Reinforced</td>
</tr>
<tr>
<td></td>
<td></td>
<td>AAC - All Aluminum Conductor</td>
</tr>
<tr>
<td></td>
<td></td>
<td>AAAC - All Aluminum Alloy Conductor</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ACAR - Aluminum Conductor Alloy Reinforced</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Alumoweld - Aluminum clad Steel Conductor</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ACSS - Aluminum Conductor Steel Supported</td>
</tr>
<tr>
<td></td>
<td></td>
<td>GTZTACSR - Gap-Type ZT Aluminum Conductor</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ACFR - Aluminum Conductor Carbon Reinforced</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ACCR - Aluminum Conductor Composite Reinforced</td>
</tr>
<tr>
<td>Insulators</td>
<td>Porcelain, Toughened glass and polymer</td>
<td>Pin Type Insulator</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Suspension Type Insulator</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Strain Type Insulator</td>
</tr>
<tr>
<td>Shield wires</td>
<td>Steel, Alumoweld and ACSR</td>
<td>Smaller Cross section compared to 3 phase conductors</td>
</tr>
<tr>
<td>Support Structures</td>
<td>Lattice steel Tower, Wood Frame</td>
<td>Wooden Poles</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Reinforced Concrete Poles</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Steel Poles</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Lattice Steel towers</td>
</tr>
</tbody>
</table>
# ACSR Conductor Characteristic

<table>
<thead>
<tr>
<th>Name</th>
<th>Overall Dia (mm)</th>
<th>DC Resistance (ohms/km)</th>
<th>Current Capacity (Amp) 75 °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mole</td>
<td>4.5</td>
<td>2.78</td>
<td>70</td>
</tr>
<tr>
<td>Squirrel</td>
<td>6.33</td>
<td>1.394</td>
<td>107</td>
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<tr>
<td>Weasel</td>
<td>7.77</td>
<td>0.9291</td>
<td>138</td>
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<tr>
<td>Rabbit</td>
<td>10.05</td>
<td>0.5524</td>
<td>190</td>
</tr>
<tr>
<td>Racoon</td>
<td>12.27</td>
<td>0.3712</td>
<td>244</td>
</tr>
<tr>
<td>Dog</td>
<td>14.15</td>
<td>0.2792</td>
<td>291</td>
</tr>
<tr>
<td>Wolf</td>
<td>18.13</td>
<td>0.1871</td>
<td>405</td>
</tr>
<tr>
<td>Lynx</td>
<td>19.53</td>
<td>0.161</td>
<td>445</td>
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<tr>
<td>Panther</td>
<td>21</td>
<td>0.139</td>
<td>487</td>
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<tr>
<td>Zebra</td>
<td>28.62</td>
<td>0.06869</td>
<td>737</td>
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<tr>
<td>Dear</td>
<td>29.89</td>
<td>0.06854</td>
<td>756</td>
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<tr>
<td>Moose</td>
<td>31.77</td>
<td>0.05596</td>
<td>836</td>
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<tr>
<td>Bersimis</td>
<td>35.04</td>
<td>0.04242</td>
<td>998</td>
</tr>
</tbody>
</table>
TL Parameters - Resistance
TL conductor resistance depends on factors such as:

- Conductor geometry
- The frequency of the AC current
- Conductor proximity to other current-carrying conductors
- Temperature
Resistance

The DC resistance of a conductor at a temperature T is given by: \[ R(T) = \rho_T \frac{l}{A} \]
where
\[ \rho_T \] = conductor resistivity at temperature T
\[ l \] = the length of the conductor
\[ A \] = the current-carry cross-sectional area of the conductor

The AC resistance of a conductor is given by: \[ R_{ac} = \frac{P_{loss}}{I^2} \]
where
\[ P_{loss} \] – real power dissipated in the conductor in watts
\[ I \] – rms conductor current
TL Conductor Resistance depends on:

➢ Spiraling:

The purpose of introducing a steel core inside the stranded aluminum conductors is to obtain a high strength-to-weight ratio. A stranded conductor offers more flexibility and easier to manufacture than a solid large conductor. However, the total resistance is increased because the outside strands are larger than the inside strands on account of the spiraling.
The layer resistance-per-length of each spirally wound conductor depends on its total length as follows:

\[ R_{\text{cond}} = \frac{\rho}{A} \sqrt{1 + \left(\pi \frac{1}{p_{\text{cond}}^2}\right)^2} \ \Omega/m \]

where

- \( R_{\text{cond}} \) - resistance of the wound conductor (\( \Omega \))
- \( \sqrt{1 + \left(\pi \frac{1}{p_{\text{cond}}^2}\right)^2} \) - Length of the wound conductor (m)
- \( \rho_{\text{cond}} = \frac{l_{\text{turn}}}{2r_{\text{laye}_r}} \) - relative pitch of the wound conductor (m)
- \( l_{\text{turn}} \) - length of one turn of the spiral (m)
- \( 2r_{\text{laye}_r} \) - Diameter of the layer (m)
Frequency:

- When voltages and currents change in time, current flow (i.e. current density) is not uniform across the diameter of a conductor.

- Thus the ‘effective’ current-carrying cross-section of a conductor with AC is less than that for DC.

- This phenomenon is known as skin effect.

- As frequency increases, the current density decreases from that at the surface of the conductor to that at the center of the conductor.
Frequency cont.

- The spatial distribution of the current density in a particular conductor is additionally altered (similar to the skin effect) due to currents in adjacent current-carrying conductors.

- This is called the ‘Proximity Effect’ and is smaller than the skin-effect.

- Using the DC resistance as a starting point, the effect of AC on the overall cable resistance may be accounted for by a correction factor $k$.

- $k$ is determined by E&M analysis. For 60 Hz, $k$ is estimated around 1.02

$$R_{AC} = k R_{DC}$$
Skin Effect

\[ \delta = \sqrt{\frac{2 \rho}{\omega \mu_r \mu_0}} \]
Temperature:

- The resistivity of conductors is a function of temperature.

- For common conductors (Al & Cu), the conductor resistance increases ~ linearly with temperature

\[ \rho(T) = \rho_0 [1 + \alpha(T - T_0)] \]

\[ \alpha \equiv \text{the temperature coefficient of resistivity} \]
TL Parameters - Conductance
Conductance

- Conductance is associated with power losses between the conductors or between the conductors and ground.

- Such power losses occur through leakage currents on insulators and via a corona.

- Leakage currents are affected by:
  - Contaminants such dirt and accumulated salt accumulated on insulators.
  - Meteorological factors such as moisture.
Conductance

- Corona loss occurs when the electric field at the surface of a conductor causes the air to ionize and thereby conduct.

- Corona loss depends on:
  - Conductor surface irregularities
  - Meteorological conditions such as humidity, fog, and rain

- Losses due to leakage currents and corona loss are often small compared to direct $I^2R$ losses on TLs and are typically neglected in power flow studies.
Corona loss
Corona loss
TL Parameters - Inductance
Inductance is defined by the ratio of the total magnetic flux flowing through (passing through) an area divided by the current producing that flux

$$\Phi = \int_S \overline{\mathbf{B}} \cdot d\mathbf{s} \propto I , \quad Observation$$

$$L = \frac{\Phi}{I} , \quad Definition$$

Inductance is solely dependent on the geometry of the arrangement and the magnetic properties (permeability $\mu$) of the medium.

Also, $B = \mu H$ where H is the mag field due to current flow only.
• Magnetic flux is the number of magnetic field lines passing through a plane area A.

• Consider an element of area dA on a surface. If the magnetic field at this element is B, then the magnetic flux through the area is \( B \cdot dA \), where dA is a vector perpendicular to the surface.

Flux is proportional to the density of flow

Flux varies by how the boundary is oriented wrt the flow

Flux is proportional to the area within the boundary
Inductance

- When a current flows through a conductor, a magnetic flux is set up which links the conductor.

- The current also establishes a magnetic field proportional to the current in the wire.

- Due to the distributed nature of a TL we are interested in the inductance per unit length (H/m).
Inductance - Examples
Inductance of a Single Wire

The inductance of a magnetic circuit that has a constant permeability ($\mu$) can be obtained as follows:

The total magnetic field $B_x$ at $x$ via Amperes law:

$$B_x = \frac{\mu_0 x}{2\pi r^2} I \ (Wb/m)$$

The flux $d\Phi$ through slice $dx$ is given by:

$$d\Phi = B_x \ da = B_x \ l \ dx \ (Wb)$$
The fraction of the current $I$ in the wire that links the area $da = dx \cdot l$ is:

$$N_f = \left(\frac{x}{r}\right)^2$$

The total flux through the wire is consequently given by:

$$\Phi = \int_0^r d\Phi = \int_0^r \left(\frac{x}{r}\right)^2 \frac{\mu_0 x I}{2\pi r^2} \, l \, dx$$

Integrating with subsequent algebra to obtain inductance-per-m yields:

$$L_{\text{wire - self}} = 0.5 \cdot 10^{-7} \ (H/m)$$
To calculate the inductance per unit length (H/m) for two-parallel wires the flux through area S must be calculated.
The inductance of a single phase two-wire line is given by:

$$L = 4 \cdot 10^{-7} \ln \left( \frac{D}{r'} \right) \text{ (H/m)}$$

where $r' = r_x' = r_y' = e^{\frac{1}{4}} r_x$
The inductance per phase of a 3-phase 3 wire line with equal spacing is given by:

\[
L_a = 2 \cdot 10^{-7} \ln \left( \frac{D}{r'} \right) \text{ (H/m) per phase}
\]
L of a Single Phase 2-conductor Line Comprised of Composite Conductors
\[ L_x = 2 \cdot 10^{-7} \ln \left( \frac{D_{xy}}{D_{xx}} \right) \]

where

\[
D_{xy} = \sqrt{\prod_{k=1}^{N} \prod_{m=1'}^{M} D_{km}} \quad \text{and} \quad D_{xx} = \sqrt{\prod_{k=1}^{N} \prod_{m=1}^{M} D_{km}}
\]

\(D_{xy}\) is referred to as GMD = geometrical mean distance between conductors x & y

\(D_{xx}\) is referred to as GMR = geometrical mean radius of conductor x
A similar $L_y$ express exists for conductor $y$.

$$L_{\text{Total}} = L_x + L_y \quad (\text{H/m})$$

Great NEWs!!!!!

Manufacturers often calculate these inductances for us for various cable arrangements 😊
Long 3-phase TLs are sometimes transposed for positive sequence balancing.

This is cleverly called ‘transposition’

The inductance (H/m) of a completely transposed three phase line may also be calculated.
\[ L_a = 2 \cdot 10^{-7} \ln \left( \frac{D_{eq}}{D_s} \right) \]

\[ D_{eq} = \sqrt[3]{D_{12}D_{13}D_{23}} \]

\( L_a \) has units of \((\text{H/m})\)

\( D_s \) is the conductor GMR for stranded conductors or \( r' \) for solid conductors
In HV TLs conductors are often arranged in one of the following 3 standard configurations.
\[ L_a = 2 \cdot 10^{-7} \ln \left( \frac{D_{eq}}{D_{SL}} \right) \]

\( L_a \) again has units of \((\text{H/m})\)

If bundle separation is large compared to the bundle size, then \(D_{eq} \sim\) the center-to-center distance of the bundles

- **2-conductor bundle:** \( D_{SL} = \frac{4}{9} \sqrt{(D_S d)^2} = \sqrt{D_S d} \)

- **3-conductor bundle:** \( D_{SL} = \frac{9}{3} \sqrt{(D_S dd)^3} = 3 \sqrt{D_S d^2} \)

- **4-conductor bundle:** \( D_{SL} = \sqrt[16]{(D_S ddd \sqrt{2})^4} = 1.091 \sqrt[4]{D_S d^3} \)
TL Parameters - Capacitance
\[ Q = CV \]

\[ C = \frac{\int_S \varepsilon \bar{E} \cdot d\bar{s}}{\int_l \bar{E} \cdot d\bar{l}} - \int_l \bar{E} \cdot d\bar{l}} \]

\[ RC = \frac{\varepsilon}{\sigma} \]
Capacitance

- A capacitor results when any two conductors are separated by an insulating medium.

- Conductors of an overhead transmission line are separated by air – which acts as an insulating medium – therefore they have capacitance.

- Due to the distributed nature of a TL we are interested in the capacitance per unit length (F/m).

- Capacitance is solely dependent on the geometry of the arrangement and the electric properties (permittivity \( \varepsilon \)) of the medium.
Calculating Capacitance

- To calculate the capacitance between conductors we must calculate two things:
  - The flux of the electric field $E$ between the two conductors
  - The voltage $V$ (electric potential) between the conductors

- Then we take the ratio of these two quantities
Since conductors in a power system are typically cylindrical, we start with this geometry for determining the flux of the electric field and potential (reference to infinity)
Gauss’s Law may be used to calculate the electric field $E$ and the potential $V$ for this geometry:

$$E_x = \frac{q}{2\pi \varepsilon x} \text{ (}V/m\text{)}$$

The potential (voltage) of the wire is found using this electric field:

$$V_{12} = \frac{q}{2\pi \varepsilon} \int_{D_1}^{D_2} \frac{dx}{x} = \frac{q}{2\pi \varepsilon} \ln \left( \frac{D_2}{D_1} \right) \text{ Volts}$$

The capacitance of various line configurations may be found using these two equations.
Capacitance - Examples
Capacitance of a single-phase 2-Wire Line

The line-to-neutral capacitance is found to be:

\[ C_n = \frac{2\pi \varepsilon}{\ln\left(\frac{D}{r}\right)} \quad (F/m) \]

The same \( C_n \) is also found for a 3-phase line-to-neutral arrangement.
The Capacitance of Stranded Conductors

e.g. a 3-phase line with 2 conductors per phase
Direct analysis yields

\[ C_{an} = \frac{2\pi \varepsilon}{\ln \left( \frac{D_{eq}}{r} \right)} \quad (F/m) \]

where

\[ D_{eq} = \sqrt[3]{D_{ab} D_{ac} D_{bc}} \]
The Capacitance of a Transposed Line

Diagram showing a transposed line with positions labeled 1, 2, and 3.
The Capacitance of a Transposed Line

\[ C_{an} = \frac{2\pi \varepsilon}{\ln \left( \frac{D_{eq}}{D_{SC}} \right)} \quad (F/m) \]

where

\[ D_{eq} = \sqrt[3]{D_{ab}D_{ac}D_{bc}} \]

2-conductor bundle: \( D_{SC} = \sqrt{r \, d} \)

3-conductor bundle: \( D_{SC} = \sqrt[3]{r \, d^2} \)

4-conductor bundle: \( D_{SC} = 1.091 \sqrt[4]{r \, d^3} \)
TLs Continued

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Annie Sebastian
Marian Mohamed
Maxwell’s Eqns. & the Telegraph Eqns.
\[ \nabla \cdot \vec{D} = \rho \]
\[ \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
Maxwell’s Equations & Telegraph Equations

- The ‘Telegrapher's Equations’ (or) ‘Telegraph Equations’ are a pair of coupled, linear differential equations that describe the voltage and current on a TL as functions of distance and time.

- The original Telegraph Equations and modern TL model was developed by Oliver Heaviside in the 1880’s.

- The telegraph equations lead to the conclusion that voltages and currents propagate on TLs as waves.
Maxwell’s Equations & Telegraph Equations

- The Transmission Line equation can be derived using Maxwell’s Equation.

- Maxwell's Equations (MEs) are a set of 4 equations that describe all we know about electromagnetics.

- MEs describe how electric and magnetic fields propagate, interact, and how they are influenced by other objects.
James Clerk Maxwell [1831-1879] was an Einstein/Newton-level genius who took the set of known experimental laws (Faraday's Law, Ampere's Law) and unified them into a symmetric coherent set of Equations known as Maxwell's Equations.

Maxwell was one of the first to determine the speed of propagation of electromagnetic (EM) waves was the same as the speed of light - and hence concluded that EM waves and visible light were really descriptions of the same phenomena.
Maxwell’s Equations – 4 Laws

**Gauss’ Law:** The total of the electric flux through a closed surface is equal to the charge enclosed divided by the permittivity. The electric flux through an area is defined as the electric field multiplied by the area of the surface projected in a plane perpendicular to the field.

\[ \nabla \cdot \vec{D} = \frac{\rho}{\varepsilon} \]

**The ‘no name’ ME:** The net magnetic flux through any closed surface is zero.

\[ \nabla \cdot \vec{B} = 0 \]
Faraday’s law: Faraday's ‘law of induction’ is a basic law of electromagnetism describing how a magnetic field interacts over a closed path (often a circuit loop) to produce an electromotive force (EMF; i.e. a voltage) via electromagnetic induction.

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

Ampere’s law: describes how a magnetic field is related to two sources; (1) the current density \( \vec{J} \), and (2) the time-rate-of-change of the displacement vector \( \vec{D} \).

\[ \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \]
The Constitutive Equations

Maxwell’s 4 equations are further augmented by the ‘constitutive equations’ that describe how materials interact with fields to relate (B & H) and (D & E).

For simple linear, isotropic, and homogeneous materials the constitutive equations are represented by:

$$\vec{B} = \mu \vec{H} \quad \mu = \mu_r \mu_o$$

$$\vec{D} = \varepsilon \vec{E} \quad \varepsilon = \varepsilon_r \varepsilon_o$$
The Telegraph Eqns.
TL Equations

- TL equations can be derived by two methods:
  - Maxwell’s equations applied to waveguides
  - Infinitesimal analysis of a section of a TL operating in ACSS with LCRG parameters (i.e. use phasors)

- The TL circuit model is an infinite sequence of 2-port components; each being an infinitesimally short segment of the TL.
The LCRG parameters of a TL are expressed as per-unit-length quantities to account for the experimentally observed distributed characteristics of the TL.

- \( R' = \) Conductor resistance per unit length, \( \Omega/m \)
- \( L' = \) Conductor inductance per unit length, \( H/m \)
- \( C' = \) Capacitance per unit length between TL conductors, \( F/m \)
- \( G' = \) Conductance per unit length through the insulating medium between TL conductors, \( S/m \)
The Infinitesimal TL Circuit
The Infinitesimal TL Circuit Model
Transmission Lines Equations

The following circuit techniques and laws are used to derive the lossless TL equations

KVL around the ‘abcd’ loop

KCL at node b

\[ V_L = L \frac{\partial i}{\partial t} \quad OLL \text{ for an inductor} \]

\[ I_C = C \frac{\partial v}{\partial t} \quad OLC \text{ for a capacitor} \]

Set \( R' = G' = 0 \) for the Lossless TL case
Apply KVL around the “abcd” loop:

\[ V(z) = V(z+\Delta z) + R' \Delta z \ I(z) + L' \Delta z \frac{\partial I(z)}{\partial t} \]

Divide all the terms by \( \Delta z \):

\[ \frac{V(z)}{\Delta z} = \frac{V(z+\Delta z)}{\Delta z} + R' I(z) + L' \frac{\partial I(z)}{\partial t} \]

As \( \Delta z \to 0 \) in the limit and \( R' = 0 \) (lossless case):

\[ \frac{\partial V(z)}{\partial z} = - \left( R' I(z) + L' \frac{\partial I(z)}{\partial t} \right) \]

\[ \frac{\partial V(z)}{\partial z} = -L' \frac{\partial I(z)}{\partial t} \]  

(Eq.1)
Apply KCL at node b:

\[ I(z) = I(z+\Delta z) + G' \Delta z \cdot V(z+\Delta z) + C' \Delta z \frac{\partial V(z+\Delta z)}{\partial t} \]

Divide all the terms by \( \Delta z \):

\[ \frac{I(z)}{\Delta z} = \frac{I(z+\Delta z)}{\Delta z} + G'V(z + \Delta z) + C' \frac{\partial V(z+\Delta z)}{\partial t} \]

As \( \Delta z \to 0 \) in the limit and \( G' = 0 \) (lossless case):

\[ \frac{\partial I(z)}{\partial z} = -\left( G'V(z) + C' \frac{\partial V(z)}{\partial t} \right) \]

\[ \frac{\partial I(z)}{\partial z} = -C' \frac{\partial V(z)}{\partial t} \]

(Eq.2)
Summarizing

(Eqn. 1) \[ \frac{\partial V(z,t)}{\partial z} = -L' \frac{\partial I(z,t)}{\partial t} \]

(Eqn. 2) \[ \frac{\partial I(z,t)}{\partial z} = -C' \frac{\partial V(z,t)}{\partial t} \]

Equations 1 & 2 are called the “Coupled First Order Telegraph Equations”
Differentiating (Eq.1) wrt ‘z’ yields:

\[ \frac{\partial^2 V(z)}{\partial z^2} = -L' \frac{\partial^2 I(z)}{\partial z \partial t} \]

(Eq.3)

Differentiating (2) wrt ‘t’ yields:

\[ \frac{\partial^2 I(z)}{\partial z \partial t} = -C' \frac{\partial^2 V(z)}{\partial t^2} \]

(Eq.4)

Plugging equation 4 into 3:

\[ \frac{\partial^2 V(z)}{\partial z^2} = L' C' \frac{\partial^2 V(z)}{\partial t^2} \]

(Eq.5)
Differentiating (Eq.1) wrt ‘t’ yields:

\[ \frac{\partial^2 V(z)}{\partial z \partial t} = - L' \frac{\partial^2 I(z)}{\partial t^2} \]  
(Eq.6)

Differentiating (2) wrt ‘z’ yields:

\[ \frac{\partial^2 I(z)}{\partial z^2} = - C' \frac{\partial^2 V(z)}{\partial z \partial t} \]  
(Eq.7)

Plugging equation 6 into 7:

\[ \frac{\partial^2 I(z)}{\partial z^2} = L' C' \frac{\partial^2 I(z)}{\partial t^2} \]  
(Eq.8)
Summarizing

(Eqn. 5) \[ \frac{\partial^2 V(z)}{\partial z^2} = L'C' \frac{\partial^2 V(z)}{\partial t^2} \]

(Eqn. 8) \[ \frac{\partial^2 I(z)}{\partial z^2} = L'C' \frac{\partial^2 I(z)}{\partial t^2} \]

Equations 5 & 8 are the v(z,t) and i(z,t) TL wave equations
What is really cool about Equations 5 & 8 is that the solution to a wave equation was already known; hence solving these equations was straightforward.

The general solution of the wave equation has the following form:

\[ f(z, t) = f(z \pm ut) \]

where the ‘wave’ propagates with Phase Velocity \( u \)

\[ u = \frac{\omega}{\beta} = \frac{\omega}{\omega \sqrt{L'C'}} = \frac{1}{\sqrt{L'C'}} \]
The solution of the general wave equation, \( f(z,t) \), describes the shape of the wave and has the following form:

\[
f(z, t) = f(z - ut)
\]

or

\[
f(z, t) = f(z + ut)
\]

The sign of \( ut \) determines the direction of wave propagation:

- \( f(z - ut) \) → Wave traveling in the \(+z\) direction
- \( f(z + ut) \) → Wave traveling in the \(−z\) direction
The ‘+z’ Traveling WE Solution

$\nabla f(z, t) = f(z - ut)$

$\nabla f(z - ut) = f(0)$

$\square z - ut = 0$

$\checkmark z = ut$
The argument of the wave equation solution is called the phase.

Hence the culmination of all of this is that, “Waves propagate by phase change”

One other key point, the form of the voltage WE and current WE are identical. Hence solutions to each must be of the same form and in phase.
Wave Equation

- $z$ increases as $t$ increases for a constant phase
TL Solutions
TL Wave Equations - Recap

(Eqn. 5) \[ \frac{\partial^2 V(z)}{\partial z^2} = L' C' \frac{\partial^2 V(z)}{\partial t^2} \]

(Eqn. 8) \[ \frac{\partial^2 I(z)}{\partial z^2} = L' C' \frac{\partial^2 I(z)}{\partial t^2} \]

Equations 5 & 8 are the \( v(z,t) \) and \( i(z,t) \) TL wave equations
ACSS TL Analysis Recipe
Initial Steps

- Typically we will know or are given the source voltage. For example, it could be:

\[ v_{ss}(t) = 10 \cos(10^7 t + 30^\circ) \text{ volts} \]

- From the time function we form the source phasor & know the frequency that the TL operates at (in ACSS operation):

\[ \omega = 10^7 \text{ s}^{-1} \]

\[ V_{ss} = 10 e^{j30^\circ} \text{ volts} \]
ACSS TL Analysis

Under ACSS conditions the voltage and current solutions to the WE may be expressed as:

\[ V(z, t) = Re\{V_0^+ e^{j(\omega t - \gamma z)} + V_0^- e^{j(\omega t + \gamma z)}\} \]

\[ V(z) = V_0^+ e^{-j\gamma z} + V_0^- e^{j\gamma z} \]  \hspace{1cm} \text{Time Harmonic Phasor Form} \]

\[ I(z, t) = Re\{I_0^+ e^{j(\omega t - \gamma z)} + I_0^- e^{j(\omega t + \gamma z)}\} \]

\[ I(z) = I_0^+ e^{-j\gamma z} + I_0^- e^{j\gamma z} = \frac{V_+}{Z_0} e^{-j\gamma z} + \frac{V_-}{Z_0} e^{j\gamma z} \]  \hspace{1cm} \text{Time Harmonic Phasor Form} \]
ACSS TL Analysis Recipe

- Step 1: Assemble $Z_0, Z_L, l, and \gamma$

- Step 2: Determine $\Gamma_L$

- Step 3: Determine $Z_{in}$

- Step 4: Determine $V_{in}$

- Step 5: Determine $V_0^+$

- Step 6: Determine $V_L$

- Step 7: Calculate $I_L$ and $P_L$
Step 1: Assemble $Z_0$, $Z_L$, $l$, $\omega$, and $\gamma$

- **Characteristic Impedance:**

$$Z_0 = \sqrt{\frac{L'}{C'}} = \frac{V_o^+}{I_o^+} = -\frac{V_o^-}{I_o^-}$$

- **Load Impedance:**

$$Z_L = \frac{V_L}{I_L} = \left(\frac{V_o^+ + V_o^-}{V_o^+ - V_o^-}\right)Z_0$$

- **Complex Propagation Constant:** ($\alpha = 0$ for lossless TL)

$$\gamma = \alpha + j\beta = j\omega\sqrt{L'C'}$$
Additional Initial Results

Phase velocity $u_P$:

$$u_P = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}$$

Identity:

$$L'C' = \mu \varepsilon$$
Step 2: Determine $\Gamma_L$

- Reflection Coefficient:
  \[
  \Gamma_L = \frac{V_0^-}{V_0^+} = -\frac{I_0^-}{I_0^+}
  \]
  \[
  \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}
  \]
  \[
  \bar{z}_L = \frac{Z_L}{Z_0}
  \]

  \[
  \Gamma_L = \frac{\bar{z}_L - 1}{\bar{z}_L + 1}
  \]

- Reflection Coefficient at distance $z$ (from load to source = -l):
  \[
  \Gamma(z) = \frac{V_0^- e^{\gamma z}}{V_0^+ e^{-\gamma z}} = \Gamma_L e^{j2\beta z}
  \]
Additional Step-2 Results

- VSWR:

\[ VSWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \]

- Range of the VSWR:

\[ 0 < VSWR < \infty \]
Step 3: Determine $Z_{in}$

- Input Impedance:
  \[ Z_{in} = \frac{V_{in}(z = -l)}{I_{in}(z = -l)} = Z_0 \frac{Z_L + Z_0 \tanh(\gamma l)}{Z_0 + Z_L \tanh(\gamma l)} \]

- For a lossless line $\rightarrow \tanh(\gamma l) = j \tan(\beta l)$:
  \[ Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \]

- At any distance $z_1$ from the load:
  \[ Z(z_1) = Z_0 \frac{Z_L + jZ_0 \tan(\beta z_1)}{Z_0 + jZ_L \tan(\beta z_1)} \]
Step 4: Determine $V_{in}$

$$V_{in} = V_{ss} \frac{Z_{in}}{Z_{in} + Z_S} = V_{in}(z = -l)$$
Step 5: Determine $V_0^+$

$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} = V_0^+ (e^{-j\beta z} + \Gamma_L e^{j\beta z})$

$V_{in} = V_{TL}(z = -l) = V_0^+ (e^{j\beta l} + \Gamma_L e^{-j\beta l})$

$V_0^+ = \frac{V_{in}}{(e^{j\beta l} + \Gamma_L e^{-j\beta l})}$
Step 6: Determine $V_L$

Voltage at the load, $V_L$:

$$V_L = V_{TL}(z = 0) = V_o^+(1 + |\Gamma_L|)$$

where:

$$V_o^+ = \frac{V_{in}}{(e^{j\beta l} + \Gamma_L e^{-j\beta l})}$$
Step 7: Calculate $I_L$ and $P_L$

$$I_L = \frac{V_L}{Z_L}$$

$$P_{avg} = P_L = \frac{1}{2} I_L^2 \text{Re}(Z_L)$$

In general, $$S_L = V_L I_L^* = P_L + j Q_L$$
Final Steps

➤ After we have determined $V_L$ in phasor form, we might be interested in the actual time function associated with $V_L$; i.e. $v_L(t)$. It is straightforward to convert the $V_L$ phasor back to the $v_L(t)$ time-domain form; e.g. suppose:

$$V_L = 5 \ e^{-j60^\circ} \ \text{volts}$$

➤ Then with

$$\omega = 10^7 \ s^{-1}$$

$$v_L(t) = 5 \ \cos(10^7 \ t - 60^\circ) \ \text{volts}$$
Surge Impedance Loading (SIL)

Lossless TL with $V_S$ fixed for line-lengths $< \lambda/4$
2-Port Analysis
Two Port Network
A two-port network is an electrical ‘black box’ with two sets of terminals.

The 2-port is sometimes referred to as a four terminal network or quadrupole network.

In a 2-port network, port 1 is often considered as an input port while port 2 is considered to be an output port.
A 2-port network model is used to mathematically analyze electrical circuits by isolating the larger circuits into smaller portions.

The 2-port works as “Black Box” with its properties specified by a input/output matrix.

Examples: Filters, Transmission Lines, Transformers, Matching Networks, Small Signal Models
Characterization of 2-Port Network

- 2-port networks are considered to be linear circuits hence the principle of superposition applies.

- The internal circuits connecting the 2-ports are assumed to be in their zero state and free of independent sources.

- 2-Port Network consist of two sets of input & output variables selected from the overall $V_1, V_2, I_1, I_2$ set:
  - 2 Independent (or) Excitation Variables
  - 2 Dependent (or) Response Variables
Characterization of 2-Port Network

Uses of 2-Port Network:

- Analysis & Synthesis of Circuits and Networks.

- Used in the field of communications, control system, power systems, and electronics for the analysis of cascaded networks.

- 2-ports also show up in geometrical optics, wave-guides, and lasers

- By knowing the matric parameters describing the 2-port network, it can be considered as black-box when embedded within a larger network.
Any linear system with 2 inputs and 2 outputs can always be expressed as:

\[
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} =
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]
<table>
<thead>
<tr>
<th>S. No.</th>
<th>Name</th>
<th>Function</th>
<th>Matrix form</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Express</td>
<td>In terms of</td>
</tr>
<tr>
<td>1.</td>
<td>Open circuit Impedance or [Z] Parameter</td>
<td>$V_1, V_2$</td>
<td>$I_1, I_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{bmatrix} V_1 \ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} &amp; Z_{12} \ Z_{21} &amp; Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \ I_2 \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>Short circuit admittance or [Y] Parameter</td>
<td>$I_1, I_2$</td>
<td>$V_1, V_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{bmatrix} I_1 \ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} &amp; Y_{12} \ Y_{21} &amp; Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \ V_2 \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>ABCD or Transmission Parameter</td>
<td>$V_1, I_1$</td>
<td>$V_2, I_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{bmatrix} V_1 \ I_1 \end{bmatrix} = \begin{bmatrix} A &amp; B \ C &amp; D \end{bmatrix} \begin{bmatrix} V_2 \ -I_2 \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>Inverse Transmission Parameter</td>
<td>$V_2, I_2$</td>
<td>$V_1, I_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{bmatrix} V_2 \ I_2 \end{bmatrix} = \begin{bmatrix} A' &amp; B' \ C' &amp; D' \end{bmatrix} \begin{bmatrix} V_1 \ -I_1 \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>Hybrid or $[h]$ Parameter</td>
<td>$V_1, I_2$</td>
<td>$I_1, V_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{bmatrix} V_1 \ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} &amp; h_{12} \ h_{21} &amp; h_{22} \end{bmatrix} \begin{bmatrix} I_1 \ V_2 \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td>Inverse hybrid or $[g]$ Parameter</td>
<td>$I_1, V_2$</td>
<td>$V_1, I_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{bmatrix} I_1 \ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} &amp; g_{12} \ g_{21} &amp; g_{22} \end{bmatrix} \begin{bmatrix} V_1 \ I_2 \end{bmatrix}$</td>
<td></td>
</tr>
</tbody>
</table>
Review

- A careful review of the preceding TL analysis reveals that all we really determined in ACSS operation was $V_{in} & I_{in}$ and $V_L & I_L$.

- This reminds us of the exact input and output form of any linear system with 2 inputs and 2 outputs which can always be expressed as:

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
Transmission Line as Two Port Network

\[ V_1 = V_{in} \]
\[ V_2 = V_L \]
\[ I_1 = I_{in} \]
\[ I_2 = I_L \]
Representation of a TL by a 2-port network

Nomenclature Used:

- $V_S$ – Sending end voltage
- $I_S$ – Sending end current
- $V_R$ – Receiving end Voltage
- $I_R$ – Receiving end current
Representation of a TL by a 2-port network

- Relation between sending end and receiving end is given as:

\[ V_S = A V_R + B I_R \]
\[ I_S = C V_R + D I_R \]

- In Matrix Form

\[
\begin{pmatrix}
V_S \\
I_S \\
\end{pmatrix}
= \begin{bmatrix}
A & B \\
C & D \\
\end{bmatrix}
\begin{pmatrix}
V_R \\
I_R \\
\end{pmatrix}
\]

\[ AD - BC = 1 \]
4 Cases of Interest

- Case 1: Short TL
- Case 2: Medium distance TL
- Case 3: Long TL
- Case 4: Lossless TL
ABCD Matrix- Short Lines

\[ I_S \quad Z = zl = (R + jL)l \quad I_R \]

\[ V_S \quad V_R \]
Short TL

- Less than ~80 Km
  - *shunt Admittance is neglected*

- Apply KVL & KCL on *abcd* loop:
  - \( V_S = V_R + Z I_R \)
  - \( I_S = I_R \)

- ABCD Matrix for short lines:
  - \[
  \begin{pmatrix}
  V_S \\
  I_S
  \end{pmatrix} =
  \begin{bmatrix}
  1 & Z \\
  0 & 1
  \end{bmatrix}
  \begin{pmatrix}
  V_R \\
  I_R
  \end{pmatrix}
  \]

- ABCD Parameters:
  - \( A = D = 1 \) per unit
  - \( B = Z \) \( \Omega \)
  - \( C = 0 \)
ABCD Matrix - Medium Lines

\[ Z = zl = (R + jL)l \]

\[ V_S \]
\[ \frac{Y}{2} \]
\[ \frac{Y}{2} \]

\[ V_R \]
Medium TL

- Ranges from ~ 80 to 250 Km
  - Nominal symmetrical-π circuit model used

- Apply KVL & KCL on $abcd$ loop:
  - $V_S = \left(1 + \frac{Z}{2}Y\right)V_R + ZI_R$
  - $I_s = Y\left(1 + \frac{YZ}{4}\right)V_R + \left(1 + \frac{YZ}{2}\right)I_R$
Medium TL

- **ABCD Matrix for medium lines:**

\[
\begin{pmatrix}
V_S \\
I_S \\
\end{pmatrix} = \begin{bmatrix}
\left(1 + \frac{YZ}{2}\right) & Z \\
Y\left(1 + \frac{YZ}{4}\right) & \left(1 + \frac{YZ}{2}\right)
\end{bmatrix}
\begin{pmatrix}
V_R \\
I_R \\
\end{pmatrix}
\]

- **ABCD Parameters:**
  - \(A = D = \left(1 + \frac{YZ}{2}\right)\) per unit
  - \(B = Z\) \(\Omega\)
  - \(C = Y\left(1 + \frac{YZ}{4}\right)\) \(\Omega^{-1}\)
ABCD Matrix - Long Lines
Long TL

- More than ~ 250 Km and beyond
  - Full TL analysis
  - $Z_C = Z_0$

- ABCD Matrix for long lines:

$$
\begin{pmatrix}
V_S \\
I_S
\end{pmatrix} =
\begin{bmatrix}
\cosh(\gamma l) & Z_C \sinh(\gamma l) \\
\frac{1}{Z_C} \sinh(\gamma l) & \cosh(\gamma l)
\end{bmatrix}
\begin{pmatrix}
V_R \\
I_R
\end{pmatrix}
$$
ABCD Matrix - Lossless Lines

- $R = G = 0$
  - $Z_0 = \sqrt{\frac{L'}{C'}}$
  - $\gamma = j\omega\sqrt{L'C'} = j\beta$

- ABCD Matrix for lossless lines:

$$
\begin{pmatrix}
V_S \\
I_S
\end{pmatrix} =
\begin{bmatrix}
\cos(\beta l) & jZ_C \sin(\beta l) \\
\frac{j}{Z_C} \sin(\beta l) & \cos(\beta l)
\end{bmatrix}
\begin{pmatrix}
V_R \\
I_R
\end{pmatrix}
$$

- ABCD Parameters:
  - $A = D = \cosh(\gamma l) = \cosh(j\beta l) = \cos(\beta l)$ per unit
  - $B = Z_0 \sinh(\gamma l) = jZ_0 \sin(\beta l)$ $\Omega$
  - $C = \frac{\sinh(\gamma l)}{Z_0} = j \frac{\sin(\beta l)}{Z_0}$ $S$
### ABCD Matrix-Summary

<table>
<thead>
<tr>
<th>Parameter</th>
<th>A = D</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Units</td>
<td>per Unit</td>
<td>Ω</td>
<td>S</td>
</tr>
<tr>
<td>Short line (less than 80 km)</td>
<td>1</td>
<td>Z</td>
<td>0</td>
</tr>
<tr>
<td>Medium line—nominal π circuit (80 to 250 km)</td>
<td>$1 + \frac{YZ}{2}$</td>
<td>Z</td>
<td>$Y \left(1 + \frac{YZ}{4}\right)$</td>
</tr>
<tr>
<td>Long line—equivalent π circuit (more than 250 km)</td>
<td>$\cosh(\gamma \ell) = 1 + \frac{Y'Z'}{2}$</td>
<td>$Z_c \sinh(\gamma \ell) = Z'$</td>
<td>$(1/Z_c) \sinh(\gamma \ell)$</td>
</tr>
<tr>
<td>Lossless line (R = G = 0)</td>
<td>$\cos(\beta \ell)$</td>
<td>$jZ_c \sin(\beta \ell)$</td>
<td>$\frac{j \sin(\beta \ell)}{Z_c}$</td>
</tr>
</tbody>
</table>
A Brief Recipe on ‘How to do’ Power Flow Analysis (PFA)

Cooking with Dr. Greg Mowry
Steps

1. Represent the power system by its one-line diagram
Steps Cont.

2. Convert all quantities to per unit (pu)

2-Base pu analysis:

\[ S_{\text{Base}} \text{ kVA, MVA, ...} \]

\[ V_{\text{Base}} \text{ V, kV, ...} \]
Steps Cont.

3. Draw the impedance diagram
Steps Cont.

4. Determine the $Y_{BUS}$

\[
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_n
\end{bmatrix}
= 
\begin{bmatrix}
Y_{11} & Y_{12} & \cdots & Y_{1n} \\
Y_{21} & Y_{22} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
Y_{n1} & \cdots & \cdots & Y_{nn}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_n
\end{bmatrix}
\]
5. Classify the Buses; aka ‘bus types’

<table>
<thead>
<tr>
<th>Bus Type</th>
<th>Given Parameters</th>
<th>Unknown Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slack Bus</td>
<td>$V, \delta$</td>
<td>$P, Q$</td>
</tr>
<tr>
<td>Generator Bus</td>
<td>$P,</td>
<td>V</td>
</tr>
<tr>
<td>Load Bus</td>
<td>$P, Q$</td>
<td>$V, \delta$</td>
</tr>
</tbody>
</table>

*Delta is the voltage angle*
Steps Cont.

6. Guess starting values for the unknown bus parameters; e.g.

   a. Slack, assume nothing
   b. Generator, assume $\delta = 0$
   c. Load, assume $V = 1$ pu, $\delta = 0$
Steps Cont.

7. Using the power flow equations find approximations for the real & reactive power using the initially guessed values and the known values for voltage/angles/admittance at the various buses. Find the difference in these calculated values with the values that were actually given; i.e. form $\Delta P$ & $\Delta Q$.

$$(\Delta \text{ value }) = (\text{Given Value}) - (\text{Approximated Value})$$

$$P_i = \sum_{i=1}^{N_{\text{bus}}} V_i V_j Y_{ij} \cos(\theta_{ij} - \delta_i + \delta_j) \quad Q_i = \sum_{i=1}^{N_{\text{bus}}} V_i V_j Y_{ij} \sin(\theta_{ij} - \delta_i + \delta_j)$$
8. Write the Jacobian Matrix for the first iteration of the Newton Raphson Method. This will have the form:

\[
[\Delta \text{values}] = [\text{Jacobian Matrix}] \times [\delta \text{ for Unknown Parameters}]
\]
\[
\begin{bmatrix}
\Delta P \\
\Delta Q \\
\Delta \delta \\
\Delta V
\end{bmatrix} = \begin{bmatrix} J \end{bmatrix} \begin{bmatrix} \Delta \delta \\
\end{bmatrix}
\]

Elements of the Jacobian matrix

\[n \neq k\]
\[
\begin{align*}
J_{1kn} &= \frac{\partial P_k}{\partial \delta_n} = V_k Y_{kn} V_n \sin(\delta_k - \delta_n - \theta_{kn}) \\
J_{2kn} &= \frac{\partial P_k}{\partial V_n} = V_k Y_{kn} \cos(\delta_k - \delta_n - \theta_{kn}) \\
J_{3kn} &= \frac{\partial Q_k}{\partial \delta_n} = -V_k Y_{kn} V_n \cos(\delta_k - \delta_n - \theta_{kn}) \\
J_{4kn} &= \frac{\partial Q_k}{\partial V_n} = V_k Y_{kn} \sin(\delta_k - \delta_n - \theta_{kn})
\end{align*}
\]

\[n = k\]
\[
\begin{align*}
J_{1kk} &= \frac{\partial P_k}{\partial \delta_k} = -V_k \sum_{n=1, n \neq k}^{N} Y_{kn} V_n \sin(\delta_k - \delta_n - \theta_{kn}) \\
J_{2kk} &= \frac{\partial P_k}{\partial V_k} = V_k Y_{kk} \cos \theta_{kk} + \sum_{n=1}^{N} Y_{kn} V_n \cos(\delta_k - \delta_n - \theta_{kn}) \\
J_{3kk} &= \frac{\partial Q_k}{\partial \delta_k} = V_k \sum_{n=1, n \neq k}^{N} Y_{kn} V_n \cos(\delta_k - \delta_n - \theta_{kn}) \\
J_{4kk} &= \frac{\partial Q_k}{\partial V_k} = -V_k Y_{kk} \sin \theta_{kk} + \sum_{n=1}^{N} Y_{kn} V_n \sin(\delta_k - \delta_n - \theta_{kn})
\end{align*}
\]

\[k, n = 2, 3, \ldots, N\]
Steps Cont.

9. Solve for the unknown differences by inverting the Jacobian and multiplying & form the next guess:

\[
\begin{bmatrix}
\Delta P \\
\Delta Q \\
\end{bmatrix} = \begin{bmatrix}
J \\
\end{bmatrix} \begin{bmatrix}
\Delta \delta \\
\Delta V \\
\end{bmatrix}
\]

\( (\text{Unknown Value})_{\text{new}} = (\text{Unknown Value})_{\text{old}} + \text{Solved (\Delta Value)} \)
Steps Cont.

10. Repeat steps 7 – 9 iteratively until an accurate value for the unknown differences as they $\to 0$. Then solve for all the other unknown parameters.

Once these iterations converge (e.g. via NR), we now have a fully solved power system with respect to $S_j$ & $V_j$ at all of the system buses!!
Power Flow Analysis (PFA)

Dr. Greg Mowry
Annie Sebastian
Marian Mohamed
Introduction
Power Flow Analysis

- In modern power systems, bus voltages and power flow are controlled.

- Power flow is proportional to $v^2$, therefore controlling both bus voltage & power flow is a nonlinear problem.

- In power engineering, the power-flow study, or load-flow study, is a numerical analysis of the flow of electric power in an interconnected system.

- A power-flow study usually a steady-state analysis that uses simplified notation such as a one-line diagram and per-unit system, and focuses on various aspects of AC power parameters, such as voltages, voltage angles, real power and reactive power.
PFA is very crucial during the power-system planning phases as well as during periods of expansion and change for meeting present and future load demands.

PFA is very useful (and efficient) at determining power flows and voltage levels under normal operating conditions.

PFA provides insight into system operation and optimization of control settings, which leads to maximum capacity at lower the operating costs.

PFA is typically performed under ACSS conditions for 3-phase power systems.
PFA Considerations

1. Generation Supplies the demand (load) plus losses.

2. Bus Voltage magnitudes remain close to rated values.

3. Gens Operate within specified real & reactive power limits.

4. Transmission lines and Transformers are not overloaded.
Load Flow Studies (LFS)

- Power-Flow or Load-Flow software (e.g. CYME, Power World, CAPE) is used for investigating power system operations.

- The SW determines the voltage magnitude and phase at each bus in a power system under balanced three-phase steady state conditions.

- The SW also computes the real and reactive power flows (P and Q respectively) for all loads, buses, as well as the equipment losses.

- The load flow is essential to decide the best operation of existing system, for planning the future expansion of the system, and designing new power system.
LFS Requirements

- Representation of the system by a one-line diagram.
- Determining the impedance diagram using information in the one-line diagram.
- Formulation of network equations & PF equations.
- Solution of these equations.
Conventional Circuit Analysis vs. PFA

- Conventional nodal analysis is not suitable for power flow studies as the input data for loads are normally given in terms of power and not impedance.

- Generators are considered power sources and not voltage or current sources.

- The power flow problem is therefore formulated as a set of non-linear algebraic equations that require numerical analysis for the solution.
Bus

- A ‘Bus’ is defined as the meeting point (or connecting point) of various components.

- Generators supply power to the buses and the loads draw power from buses.

- In a power system network, buses are considered as nodes and hence the voltage is specified at each bus.
The Basic 2-Bus Analysis
The Foundational Heart of PFA
Quick Review
Motivation

- The basic 2-Bus analysis must be understood by all power engineers.

- The 2-Bus analysis is effectively the ‘Thevenin Equivalent’ of all power systems and power flow analysis.

- Many variant exist, by my count, at least a dozen. We will explore the basic 2-Bus PFA that models transmission systems.
The Basic 2-Bus Power System

\[ V_S = V_S \angle \delta = V_R e^{j\delta} \quad \bar{V}_R = V_R \angle 0^\circ = V_R \]

\[ S = \text{‘Send’} \quad R = \text{‘Receive’} \]
The Basic 2-Bus Power System

\[
\begin{align*}
\bar{V}_S &= V_S \angle \delta = V_R e^{j\delta} \\
\bar{V}_R &= V_R \angle 0^\circ = V_R
\end{align*}
\]

S = ‘Send’ \quad R = ‘Receive’
2-Bus Phasor Diagram

\[ V_R \equiv \text{the Reference} \]
Discussion

1) $V_S$ leads $V_R$ by $\delta$

2) KVL $\Rightarrow V_S - jX I = V_R$

\[
\therefore \quad I = \frac{V_S - V_R}{jX}
\]
3) At the receive end,

\[ S_R = P_R + jQ_R = V_R I^* \]

\[ S_R = V_R \left[ \frac{V_S \angle \delta - V_R}{jX} \right]^* \]
Discussion

4) Therefore the complex power received is (after algebra):

\[ S_R = \frac{V_S V_R}{X} \sin(\delta) + j \left[ \frac{V_S V_R}{X} \cos(\delta) - \frac{V_R^2}{X} \right] \]

\[ S_R = P_R + jQ_R \]

Recall \[ \bar{V}_S = V_S \angle \delta = V_S e^{j\delta} = V_S (\cos \delta + j \sin \delta) \]
Discussion

5) Or:

\[ P_R = \frac{V_S V_R}{X} \sin(\delta) \]

\[ Q_R = \left[ \frac{V_S V_R}{X} \cos(\delta) - \frac{V_R^2}{X} \right] \]
Discussion

6) Note that if $V_S \sim V_R \sim V$ then:

\[ P_R \approx \frac{V^2}{X} \delta \]

\[ Q_R \approx \frac{V}{X} \Delta V \]
2-Bus, Y, and PFA Lead-in
Two Bus System – arbitrary phase angles

\[ V_s < \phi_1 \quad \text{Z} \quad V_T < \phi_2 \]

Bus 1 \quad \Rightarrow \quad \text{I} \quad \Rightarrow \quad \text{Bus 2}
Two Bus System

➢ The complex AC power flow at the receiving end bus can be calculated as follows:

\[ S_r = V_r I^* \]

➢ Similarly, the power flow at the sending end bus is:

\[ S_s = V_s I^* \]

where

\[ \overline{V_s} = V_s e^{j\phi_1} \] is the voltage and phase angle at the sending bus.

\[ \overline{V_r} = V_r e^{j\phi_2} \] is the voltage and phase angle at the receiving bus.

Z is the complex impedance of the TL.
Two Bus System

\[ I = \frac{V_s - V_r}{Z} \]

- In power flow calculations, it is more convenient to use admittances rather than impedances

\[ Y = Z^{-1} \]

\[ I = Y (V_s - V_r) \]

- This 2-bus equation forms the basis of PF calculations.
The Admittance Matrix

- The use of admittances greatly simplifies PFA

- e.g. an open circuit between two buses (i.e. no connection) results in $Y = 0$ (which is computationally easy to deal with) rather than $Z = \infty$ (which is computationally hard to deal with).

- $V_{ij}$ is the voltage at bus i wrt bus j; bus i positive, bus j negative

- $I_{ab}$ represents the current flow from bus a to bus b

- Recall that a ‘node voltage’ (bus voltage) is the voltage at that node (Bus) with respect to the reference.
Admittance Matrix Method (1)

- NV analysis results in n-1 node equations (bus equations) which is analytically very efficient for systems with large n.

- Given systems with n large, an efficient and automatable method is needed/required for PFA. NVM and Y-matrix methods are so.

- In circuit analysis, series circuit elements are typically combined to form a single impedance to simplify the analysis. This is generally not done in PFA.
Admittance Matrix Method (2)

- In PFA, combining series elements is not typically done as we seek to keep track of the impedances of various assets and their power flows; e.g. in:
  i. Generators
  ii. TLs
  iii. XFs
  iv. FACTs

- Voltages sources (e.g. generators) have an associated impedance (the Thevenin impedance) that is typically inductive in ACSS
Y-Matrix Example
Example Circuit
Step 1: Identify nodes (buses), label them, and select one of them as the reference or ground.
Step 2 & 3: (2) Replace voltage sources and their Thevenin impedance with their Norton equivalent. (3) Replace impedance with their associated admittances.
Step 4: Form the system matrix

\[ Y_{Bus} * V = I \]  

(Eqn-1)

where,

\[ Y_{bus} = \text{Bus Admittance matrix of order (n x n); known} \]
\[ V = \text{Bus (node) voltage matrix of order (n x 1); unknown} \]
\[ I = \text{Source current matrix of order (n x 1); known} \]
Step 5: Solve the system matrix for bus voltages

\[ V = Y_{Bus}^{-1} \times I \quad (Eqn-2) \]

Note: at this point the various known bus voltages (amplitude and phase) may be used to calculate currents and power flow.
Stuffing the Y-Matrix

- Y-matrix elements are designated as $Y_{kn}$ where $k$ refers to the $k^{th}$ bus and $n$ refers to the $n^{th}$ bus.

- Diagonal elements $Y_{kk}$:
  - $Y_{kk} = \text{sum of all of the admittances connected to the } k^{th} \text{ bus}$

- Off-diagonal elements $Y_{kn}$:
  - $Y_{kn} = Y_{nk} = -(\text{sum of the admittances in the branch connecting bus } k \text{ to bus } n)$
Y-Matrix Terminology

- $Y_{kk}$, the main diagonal elements of $Y$, are referred to as the:
  - “Self-admittance”; or
  - “Driving point admittance”
  of bus $k$

- $Y_{kn}$, the off diagonal elements of $Y$, are referred to as the:
  - “Mutual-admittance”; or
  - “Transfer admittance”
  between bus $k$ and bus $n$
Y-Matrix Notes

- If there is no connection between bus k and bus n, then $Y_{kn} = 0$ in the Y-matrix; hence there are 2 zeros in the Y-matrix for this situation since $Y_{kn} = Y_{nk}$.

- If a new connection is made between 2 buses k & n in a network, then only 4 elements of the Y matrix change, all of the others remain the same. The elements that change are:
  - $Y_{kk}$
  - $Y_{nn}$
  - $Y_{kn} = Y_{nk}$
The ‘stuffed’ Y-Matrix

\[
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4
\end{bmatrix} =
\begin{bmatrix}
Y_{11} & Y_{12} & Y_{13} & 0 \\
Y_{21} & Y_{22} & Y_{23} & 0 \\
Y_{31} & Y_{32} & Y_{33} & Y_{34} \\
0 & 0 & Y_{43} & Y_{44}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix}
\]

Note that since no direct connections exist between Bus 1 & 4 or Bus 2 & 4, the corresponding entries in the Y matrix are zero.
Y-Matrix Math
PSC Reminder

For a passive network element(s), the PSC yields:

\[ I = \frac{V_1 - V_2}{Z} \]

\[ I = Y (V_1 - V_2) \]
Consider the following bus in a network:
Y-Matrix Equations

KCL at bus 3 yields:

\[ I_3 = I_b + I_d + I_a + I_c \]

\[ I_3 = \frac{V_3}{Z_{C1}} + \frac{V_3}{Z_{C2}} + \frac{V_3 - V_1}{Z_{13}} + \frac{V_3 - V_2}{Z_{23}} \]

\[ I_3 = V_3 Y_{3,\text{total to gnd}} + \sum_{m \neq 3} \frac{V_3 - V_m}{Z_{m3}} \]
Y-Matrix Equations

Algebra yields:

\[ I_3 = V_3 \ Y_{3,\text{total to gnd}} + \sum_{m \neq 3} \frac{V_3 - V_m}{Z_{m3}} \]

\[ I_3 = V_3 \ Y_{3,\text{total to gnd}} + \sum_{m \neq 3} Y_{m3} (V_3 - V_m) \]

\[ I_3 = V_3 \left( Y_{3,\text{total to gnd}} + \sum_{m \neq 3} Y_{m3} \right) - \sum_{m \neq 3} Y_{m3} V_m \]
Y-Matrix Equations

Finally:

\[ I_3 = V_3 \left( Y_{3,total to gnd} + \sum_{m \neq 3} Y_{m3} \right) - \sum_{m \neq 3} Y_{m3} V_m \]

Or in general

\[ I_k = V_k \left( Y_{k,total to gnd} + \sum_{m \neq k} Y_{mk} \right) - \sum_{m \neq k} Y_{mk} V_m \]
Hence:

\[ I_k = V_k \left( Y_{k, \text{total to gnd}} + \sum_{m \neq k} Y_{mk} \right) - \sum_{m \neq k} Y_{mk} V_m \]

Admittance connected to the \( k^{\text{th}} \) bus

Admittance between the \( k^{\text{th}} \) and \( m^{\text{th}} \) bus

(i)  
(ii)
Y-Matrix Equations

In terms of Y-matrix elements:

\[ Y_{kk} = \left( Y_{k, \text{total to gnd}} + \sum_{m \neq k} Y_{mk} \right) \]  \hspace{1cm} (i)

\[ Y_{km} = Y_{mk} = -\frac{1}{Z_{km}} \]  \hspace{1cm} (ii)
Y-Matrix Equations

Using the short-hand notation:

\[ I_k = V_k Y_{kk} + \sum_{m \neq k} V_m Y_{mk} \]

\[ Y_{km} = Y_{mk} \]
Consider a 3-Bus System
Let’s focus on Bus 2

\[ I_2 = V_2 Y_{22} + V_1 Y_{21} + V_3 Y_{23} \]

\[ I_2 = V_1 Y_{21} + V_2 Y_{22} + V_3 Y_{23} \]

\[
\begin{bmatrix}
I_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
Y_{21} & Y_{22} & Y_{23}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
\]
The General Case

For a n-Bus system we have:

\[
\begin{bmatrix}
I_1 \\
\vdots \\
I_n
\end{bmatrix} =
\begin{bmatrix}
Y_{11} & \cdots & Y_{1n} \\
\vdots & \ddots & \vdots \\
Y_{n1} & \cdots & Y_{nn}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
\vdots \\
V_n
\end{bmatrix}
\]

Known sources  The Y-Matrix  To be calculated
Power Flow Equations
The current at bus \( k \) in an \( n \)-bus system is:

\[
\vec{I}_k = Y_{k1}\vec{V}_1 + Y_{k2}\vec{V}_2 + \cdots + Y_{kn}\vec{V}_n
\]

where

\[
\vec{V}_k = V_k e^{j\theta_k} \\
\vec{V}_m = V_m e^{j\theta_m} \\
Y_{km} = Y_{km} e^{j\delta_{km}}
\]
The complex power $S$ at bus $k$ is given by:

$$S_k = V_k I_k^* = P_k + jQ_k$$

$$S_k = V_k \left[ \sum_{m=1}^{n} Y_{km} V_m \right]^*$$
Substituting the voltage and admittance phasors into the complex power equations yields:

\[ S_k = P_k + jQ_k = V_k e^{j\theta_k} \left[ \sum_{m=1}^{n} Y_{km} e^{j\delta_{km}} V_m e^{j\theta_m} \right]^* \]

By Euler’s Theorem

\[ \overline{V_k V_m}^* = V_k V_m \left[ \cos(\theta_{km}) + j \sin(\theta_{km}) \right] \]

where

\[ \theta_{km} = \theta_k - \theta_m \]
Expanding the complex power summation into real and imaginary parts and associating these with P & Q respectively yields our fundamental PFA equations:

\[
P_k = \sum_{m=1}^{n} V_k V_m Y_{km} \cos(\theta_k - \theta_m + \delta_{km})
\]

\[
Q_k = \sum_{m=1}^{n} V_k V_m Y_{km} \sin(\theta_k - \theta_m + \delta_{km})
\]
Key **SUMMARY** points:

- The bus voltage equations, $I=YV$, are linear

- The power flow equations are quadratic, $V^2$ appears in all $P$ & $Q$ terms

- Consequently simultaneously controlling bus voltage and power flow is a non-linear problem

- In general this forces us to use numerical solution methods for solving the power flow equations while controlling bus voltages
Network Terminology
Simple 3-Bus Power System

1. Slack Bus
2. PQ Bus
3. PV Bus

Z_{12}
Z_{13}
Z_{23}
# Bus Types

<table>
<thead>
<tr>
<th>Bus Type</th>
<th>Total number of buses</th>
<th>Quantities Specified</th>
<th>Quantities to be obtained</th>
</tr>
</thead>
<tbody>
<tr>
<td>Load Bus (or) PQ Bus</td>
<td>n-m</td>
<td>P, Q</td>
<td></td>
</tr>
<tr>
<td>Generator Bus (or) Voltage Controlled Bus (or) PV Bus</td>
<td>m-1</td>
<td>P,</td>
<td>V</td>
</tr>
<tr>
<td>Slack Bus (or) Swing Bus (or) Reference Bus</td>
<td>1</td>
<td></td>
<td>V</td>
</tr>
</tbody>
</table>
Bus Terminology

- \( P = P_G - P_D \)
- \( Q = Q_G - Q_D \)

- \( P_G \) & \( Q_G \) are the Real and Reactive Power supplied by the power system generators connected

- \( P_D \) & \( Q_D \) are respectively the Real and Reactive Power drawn by bus loads

- \( V \) = Magnitude of Bus Voltage
- \( \delta \) = Phase angle of Bus Voltage
The Need for a Slack Bus

- TL losses can be estimated only if the real and reactive power at all buses are known.

- The powers in the buses will be known only after solving the load flow equations.

- For these reasons, the real and reactive power of one of the generator bus is not specified.

- This special bus is called Slack Bus and serves as the system reference and power balancing agent.
The Need for a Slack Bus

➢ It is assumed that the Slack Bus Generates the real and reactive power of the Transmission Line losses.

➢ Hence for a Slack Bus, the magnitude and phase of bus voltage are specified and real and reactive powers are obtained through the load flow equation.

\[
\begin{align*}
\left( \frac{\text{Sum of Complex Power of Generator}}{\text{Sum of Complex Power of Loads}} \right) &= \left( \frac{\text{Sum of Complex Power of Loads}}{\text{Total (Complex)Power loss in Transmission Lines}} \right) \\
\end{align*}
\]

or

\[
\begin{align*}
\left( \frac{\text{Total (Complex) power loss in Transmission Lines}}{\text{Power of Loads}} \right) &= \left( \frac{\text{Sum of Complex Power of Loads}}{\text{Power of Generator}} \right) \\
\end{align*}
\]
Solving the PF equations
There are several methods available for solving the power flow equations:

- Direct solutions to linear algebraic equations
  - Gauss Elimination

- Iterative solutions to linear algebraic equations
  - Jacobi method
  - Gauss-Siedel method

- Iterative solutions to nonlinear algebraic equations
  - Newton-Raphson method
Gauss-Seidel Method

In numerical linear algebra, the Gauss–Seidel method, also known as the Liebmann method or the method of successive displacement, is an iterative method used to solve a linear system of equations. It is named after the German mathematicians Carl Friedrich Gauss and Philipp Ludwig von Seidel, and is similar to the Jacobi method. Though it can be applied to any matrix with non-zero elements on the diagonals, convergence is only guaranteed if the matrix is either diagonally dominant, or symmetric and positive definite. It was only mentioned in a private letter from Gauss to his student Gerling in 1823.
Carl Gustav Jacob Jacobi was a German mathematician, who made fundamental contributions to elliptic functions, dynamics, differential equations, and number theory.
Newton-Raphson Method
In numerical analysis, Newton's method (also known as the Newton–Raphson method), is named after Isaac Newton and Joseph Raphson. It is a method for finding successively better approximations to the roots (or zeroes) of a real-valued function.

\[ x; f(x) = 0 \]

The Newton–Raphson method in one variable is implemented as follows:

- The method starts with a function \( f \) defined over the real numbers \( x \)
- The function's derivative \( f' \)
- An initial guess \( x_0 \) for a root of the function \( f \).

If the function is ‘well behaved’ then a better approximation to the actual root is \( x_1 \):

\[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \]
Newton-Raphson Method

The Newton-Raphson method is a very popular non-linear equation solver as:

- NR is very powerful and typically converges very quickly
- Almost all PFA SW uses Newton-Raphson
- Very forgiving when errors are made in one iterations; the final solution will be very close to the actual root
Newton-Raphson Method

Procedure:

**Step 1:** Assume a vector of initial guess $x(0)$ and set iteration counter $k = 0$

**Step 2:** Compute $f_1(x^{(k)})$, $f_2(x^{(k)})$, …… $f_n(x^{(k)})$

**Step 3:** Compute $\Delta m_1$, $\Delta m_2$, …… $\Delta m_n$.

**Step 4:** Compute error $= \max[|\Delta m_1|, |\Delta m_2|, \ldots \ldots |\Delta m_n|]$

**Step 5:** If error $\leq \varepsilon$ (pre-specified tolerance), then the final solution vector is $x(k)$ and print the results. Otherwise go to step 6

**Step 6:** Form the Jacobin matrix analytically and evaluate it at $x = x^{(k)}$

**Step 7:** Calculate the correction vector $\Delta x = [\Delta m_1, \Delta m_2, \ldots \ldots \Delta m_n]^T$ by using equation d

**Step 8:** Update the solution vector $x^{(k+1)} = x^{(k)} + \Delta x$ and update $k = k+1$ and go back to step 2.
Consider a 2-D example where in general there are n equations are given for the n unknown quantities $x_1, x_2, x_3 \cdots x_n$.

\[
\begin{align*}
f_1(x_1, x_2) &= k_1 \\
f_2(x_1, x_2) &= k_2
\end{align*}
\]  

(Equation set $a$)

- $k_1$ and $k_2$ are constants

- Let’s assume that

\[
\begin{align*}
x_1 &\sim [x_1^{(0)} + \Delta x_1^{(0)}] \\
x_2 &\sim [x_2^{(0)} + \Delta x_2^{(0)}]
\end{align*}
\]

- Then

\[
\begin{align*}
k_1 &= f_1(x_1^{(0)} + \Delta x_1^{(0)}, x_2^{(0)} + \Delta x_2^{(0)}) \\
k_2 &= f_1(x_1^{(0)} + \Delta x_1^{(0)}, x_2^{(0)} + \Delta x_2^{(0)})
\end{align*}
\]
Newton-Raphson Method

- Applying the first order Taylor Expansion to \( f_1 \) and \( f_2 \)

\[
\begin{align*}
k_1 & \cong f_1(x_1^{(0)}, x_2^{(0)}) + \Delta x_1^{(0)} \frac{\partial f_1}{\partial x_1} + \Delta x_2^{(0)} \frac{\partial f_1}{\partial x_2} \\
k_2 & \cong f_2(x_1^{(0)}, x_2^{(0)}) + \Delta x_1^{(0)} \frac{\partial f_2}{\partial x_1} + \Delta x_2^{(0)} \frac{\partial f_2}{\partial x_2}
\end{align*}
\]

- ‘Equation set b’ expressed in matrix form,

\[
\begin{bmatrix}
k_1 - f_1(x_1^{(0)}, x_2^{(0)}) \\
k_2 - f_2(x_1^{(0)}, x_2^{(0)})
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix}
\begin{bmatrix}
\Delta x_1^{(0)} \\
\Delta x_2^{(0)}
\end{bmatrix}
\]

(Equation c)
Then

\[
\begin{bmatrix}
\Delta k_1^{(0)} \\
\Delta k_2^{(0)}
\end{bmatrix} = J^{(0)} \begin{bmatrix}
\Delta x_1^{(0)} \\
\Delta x_2^{(0)}
\end{bmatrix}
\]

Adjusting the guess according to

\[
x_1^{(1)} = [x_1^{(0)} + \Delta x_1^{(0)}]
\]

\[
x_2^{(1)} = [x_2^{(0)} + \Delta x_2^{(0)}]
\]

Iterate until \(|\Delta x_i^{(N)}| < \varepsilon\)
Newton-Raphson Method

For \( n \) equations in \( n \) unknown quantities \( x_1, x_2, x_3 \ldots x_n \).

\[
\begin{align*}
    f_1(x_1, x_2 \ldots \ldots x_n) &= b_1 \\
    f_2(x_1, x_2 \ldots \ldots x_n) &= b_2 \\
    \vdots \\
    f_n(x_1, x_2, \ldots \ldots x_n) &= b_n
\end{align*}
\]

(Equation set a)
In ‘equation set a’ the quantities \( b_1, b_2, \cdots b_n \) as well as the functions \( f_1, f_2, \cdots \cdots f_n \) are known.

To solve the equation set an initial guess is made; the initial guesses be denoted as \( x_1(0), x_2(0), \cdots x_n(0) \).

First order Taylor’s series expansions (neglecting the higher order terms) are carried out for these equation about the initial guess.

The vector of initial guesses is denoted as \( x^{(0)} = [x_1^{(0)}, x_2^{(0)}, \cdots x_n^{(0)}]^T \).

Applying Taylor’s expansion to the “equation set a” will give:
\[ f_1(x_1^{(0)}, x_2^{(0)} \ldots \ldots x_n^{(0)}) + \frac{\partial f_1}{\partial x_1} \Delta x_1 + \frac{\partial f_1}{\partial x_2} \Delta x_2 + \ldots + \frac{\partial f_1}{\partial x_n} \Delta x_n = b_1 \]
\[ f_2(x_1^{(0)}, x_2^{(0)} \ldots \ldots x_n^{(0)}) + \frac{\partial f_2}{\partial x_1} \Delta x_1 + \frac{\partial f_2}{\partial x_2} \Delta x_2 + \ldots + \frac{\partial f_2}{\partial x_n} \Delta x_n = b_2 \]
\[ \vdots \]
\[ f_n(x_1^{(0)}, x_2^{(0)} \ldots \ldots x_n^{(0)}) + \frac{\partial f_n}{\partial x_1} \Delta x_1 + \frac{\partial f_n}{\partial x_2} \Delta x_2 + \ldots + \frac{\partial f_n}{\partial x_n} \Delta x_n = b_n \]

‘Equation set b’ can be written in matrix form as:

\[
\begin{bmatrix} f_1(x^{(0)}) \\ f_2(x^{(0)}) \\ \vdots \\ f_n(x^{(0)}) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \ldots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
\]
Hence ‘equation c’ can be written as

\[
\begin{bmatrix}
\Delta x_1 \\
\Delta x_2 \\
\vdots \\
\Delta x_n
\end{bmatrix} = [J^{-1}]
\begin{bmatrix}
b_1 - f_1(x^{(0)}) \\
b_2 - f_2(x^{(0)}) \\
\vdots \\
b_n - f_n(x^{(0)})
\end{bmatrix} = [J^{-1}]
\begin{bmatrix}
\Delta m_1 \\
\Delta m_2 \\
\vdots \\
\Delta m_n
\end{bmatrix}
\]

(Equation d)

‘Equation d’ is the basic equation for solving the n algebraic equations given in ‘equation set a’.
NR Method Flowchart

1. Start
   - Read number of buses n. Read line admittances, \( y_{ij} \). Read slack bus voltage. For load buses \( P_g \) & \( Q_g \). For generator buses read \( P_{g,\text{spec}} \), \( Q_{g,\text{max}} \), \( Q_{g,\text{min}} \)
   - Form \( Y_{bus} \) matrix and determine \( G_{jk} \) & \( B_{jk} \) for \( j=1,2,...,n \) and \( k=1,2,...,n \)
   - Set \( V_{p,0} = 1+j0.0 \) except slack bus. Calculate \( e_p^0 \) & \( f_p^0 \) for \( p=1,2,n \). Set Convergence criterion. Set iteration count, \( k=0 \)

2. Set bus count, \( p=1 \)
   - Check for Slack Bus
     - Yes → Calculate \( P_p^0 \), \( Q_p^0 \) and \( P_p^0 \)
     - No → Check for Generator Bus
       - Yes → Set \( Q_{p,\text{spec}} = Q_{p,\text{min}} \)
       - No → Check if \( Q_p = Q_{p,\min} \)
       - Yes → Set \( Q_{p,\text{spec}} = Q_{p,\max} \)
       - No → Check if \( Q_p = Q_{p,\max} \)
       - Yes → Calculate \( \Delta Q_p \)
       - No → Calculate \( I\Delta V_p \)

3. Advance bus count \( p = p+1 \)
   - Check if \( p > n \)
     - Yes → 1
     - No → Check for Generator Bus
       - Yes → \( e_{p+1}^k = |IV_{p+1}^k|\text{spec cos } \delta_{p+1}^k \)
       - No → \( f_{p+1}^k = |IV_{p+1}^k|\text{spec sin } \delta_{p+1}^k \)

4. Calculate the largest absolute value of residue, \( \Delta E \)
   - Check for \( \Delta E < \epsilon \)
     - Yes → Stop
     - No → Determine the elements of Jacobian Matrix (J)
       - Calculate the line flows and slack bus power
         - \( \Delta e_p \)
         - \( \Delta f_p \)
         - Calculate \( IV_{p+1}^k \) and \( \delta_{p+1}^k \)

5. Check for Generator Bus
Power Flow Example
(using Power World)
Power Flow Example

\[ S_{Base} = 100 \text{MVA} \]
Step 1: Calculate Y-Bus:

\[ Y_{Bus} = \begin{bmatrix} -j10 & j10 & 0 \\ j10 & -j30 & j20 \\ 0 & j20 & -j20 \end{bmatrix} \]
**Step 2:** Define Bus types and their quantities:

<table>
<thead>
<tr>
<th>Bus No</th>
<th>Bus Type</th>
<th>Quantities Specified</th>
<th>Quantities to be obtained</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>Slack Bus</td>
<td>$V_1, \delta_1$</td>
<td>---</td>
</tr>
<tr>
<td>Two</td>
<td>Generator Bus</td>
<td>$P_2, V_2, \delta_2, Q_2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>PV</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Three</td>
<td>Load Bus</td>
<td>$P_3, Q_3$</td>
<td>$V_3, \delta_3$</td>
</tr>
<tr>
<td></td>
<td>PQ</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Step 3: User Power Equations to solve for unknowns:

\[
P_2 = \sum_{m=1}^{3} V_2 V_m Y_{2m} \cos(\theta_{2k} - \delta_2 + \delta_m)
\]

\[
P_3 = \sum_{m=1}^{3} V_3 V_m Y_{3m} \cos(\theta_{3k} - \delta_3 + \delta_m)
\]

\[
Q_3 = -\sum_{m=1}^{3} V_3 V_m Y_{3m} \sin(\theta_{3k} - \delta_3 + \delta_m)
\]
Power Flow Example, cont.

1 = 10 \sin \delta_2 - 20 \ V_3 \ \sin(-\delta_2 + \delta_3) \quad \text{Eq.1}

2 = 20 \ V_3 \ \sin(-\delta_3 + \delta_2) \quad \text{Eq.2}

0.5 = 20 \ V_3 \ \cos(-\delta_3 + \delta_2) - 20 \ V_3^2 \quad \text{Eq.3}
Power Flow Example, cont.

**Step 4:** Apply Newton-Raphson on Eq.2 and Eq.3 to find $V_3$ and $(-\delta_3 + \delta_2)$; then on Eq.1 to find $\delta_2$ followed by $\delta_3$

$$V_3 = 0.9689 \text{ pu}$$

$$\delta_2 = -5.739 \text{ Deg}$$

$$\delta_3 = -11.664 \text{ Deg}$$
Step 5: Use $V_3$, $\delta_2$, and $\delta_3$ to find $Q_2$ and $Q_1$:

\[ Q_2 = - \sum_{m=1}^{3} V_2 V_m Y_{2m} \sin(\theta_{2k} - \delta_2 + \delta_m) \]

\[ Q_1 = - \sum_{m=1}^{3} V_1 V_m Y_{1m} \sin(\theta_{1k} - \delta_1 + \delta_m) \]

$Q_2 = 0.776$ pu

$Q_2 = 0.776$ pu
## Power Flow Example, cont.

### Final Soliton

<table>
<thead>
<tr>
<th>Bus No</th>
<th>V</th>
<th>$\delta$</th>
<th>P</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>1</td>
<td>0°</td>
<td>1</td>
<td>0.05</td>
</tr>
<tr>
<td>Two</td>
<td>1</td>
<td>$-5.7392^o$</td>
<td>1</td>
<td>0.776</td>
</tr>
<tr>
<td>Three</td>
<td>0.9689</td>
<td>$-11.664^o$</td>
<td>$-2$</td>
<td>$-0.5$</td>
</tr>
</tbody>
</table>
Power World Example
Power World Example, cont.

Y-Bus

![Y-Bus diagram](image-url)
Power World Example, cont.

Jacobin Matrix
### Power World Example, cont.

#### Power Flow Soliton

**Power Flow Tools**

- Single Solution - Full Newton
- Simulator Options...
- Contingency Analysis...
- RAS + CTG Case Info...
- Sensitivities
- Time Step Simulation...
- Limit Monitoring...
- Difference Flows...
- Scale Case...
- Model Explorer...
- Other Tools

**Buses**

<table>
<thead>
<tr>
<th>Number</th>
<th>Name</th>
<th>Area Name</th>
<th>Nom KV</th>
<th>PU Volt</th>
<th>Volt (KV)</th>
<th>Angle (Deg)</th>
<th>Load MW</th>
<th>Load Mvar</th>
<th>Gen MW</th>
<th>Gen Mvar</th>
<th>Switched Shunts Mvar</th>
<th>Act G Shunt MW</th>
<th>Act B Shunt Mvar</th>
<th>Area Num</th>
<th>Zone Num</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Bus 1</td>
<td>1</td>
<td>138.00</td>
<td>1.00000</td>
<td>138.000</td>
<td>0.0000</td>
<td>100.00</td>
<td>5.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>Bus 2 (PV)</td>
<td>1</td>
<td>138.00</td>
<td>1.00000</td>
<td>138.000</td>
<td>-5.7392</td>
<td>100.00</td>
<td>77.65</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>Bus 3 (PQ)</td>
<td>1</td>
<td>138.00</td>
<td>0.96886</td>
<td>133.702</td>
<td>-11.6633</td>
<td>200.00</td>
<td>50.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>